ON THE STABILITY OF MOTION OF A GYROSCOPE ON GIMBALS

(OB USTOICHIVOSTI DVIZHENIIA GIROSKOPA V KARDANOVOM PODVESE)

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Nikolai [1], in the year 1939, investigated an inertially balanced symmetrical gyroscope on gimbals, taking into account masses of the gimbal rings. He discovered the interesting fact, that the stability of the gyroscope axis in vertical position depends on both the magnitude and direction of the angular velocity vector of the outer ring.

Magnus [2] proved the occurrence of a similar phenomenon for an unbalanced symmetrical gyroscope on gimbals. He constructed the Liapunov function using the Chetaev method and obtained sufficient conditions for the stability of rotation of a gyroscope about the vertical axis of the outer gimbal ring.

Skimel* investigated the stability of a regular precession of a gyroscope on gimbals when the nutation angle $\theta \neq 0$.

In this paper the author constructs the Liapunov function in the form of a linear combination of the first integrals of the equations of motion and derives from it the sufficient conditions for the stability of motion of the regular precession of a gyroscope on gimbals, which yields as a special case (when $\theta_0 = 0$) the necessary condition for stability of the vertical position of the gyroscope axis. The author investigates also the influence of the dissipative forces on the stability of motion of a gyroscope.

1. Let us consider the motion of a symmetrical gyroscope on gimbals,

* V.N. Skimel, Nekotorye zadachi dvizhenia i ustoichivosti tiazhelogo giroskopa. Avtoreferat kandidatskoi dissertatsi, (Certain problems of motion and stability of a heavy gyroscope. Dissertation for candidate's degree), Kazan, 1955.

taking into account the masses of the gimbal rings. The stationary axis of rotation of the outer gimbal ring is vertical, the axis of rotation of the inner ring is horizontal and the centers of gravity of the gyroscope and of the inner ring are on the axis of symmetry of the gyroscope.

We shall introduce two right-handed orthogonal coordinate systems, a fixed one $O\xi\eta\zeta$, and a moving one $Oxyz$, whose origins coincide with the stationary point O in the gyroscope system. The axis $O\zeta$ of the fixed coordinate system is vertical and coincides with the axis of rotation of the outer ring; the axes $O\xi$ and $O\eta$ are in a horizontal plane. The coordinate system $Oxyz$ moves with the inner gimbal ring, the Ox and Oz axes coincide with the axis of rotation of the inner ring and with the axis of symmetry of the gyroscope, respectively, and the Oy axis is perpendicular to the middle plane of the inner ring.

The orientation of the whole gyroscope system in the space $O\xi\eta\zeta$ can be represented by the three Eulerian angles; the angle of nutation θ, the angle of precession ψ and the angle ϕ which is the angle of rotation of the gyroscope itself with respect to the coordinate system $Oxyz$. The projections of the instantaneous angular velocity of the gyroscope, ω, and of the inner ring, ω_1, on the coordinate axes Ox, Oy, Oz, are expresse in terms of the Eulerian angles and their time derivatives as follows:

$$\begin{aligned} p &= \theta', \qquad & q &= \psi' \sin\theta, \qquad & r &= \varphi' + \psi' \cos\theta \\ p_1 &= \theta', \qquad & q_1 &= \psi' \sin\theta, \qquad & r_1 &= \psi' \cos\theta \end{aligned} \tag{1.1}$$

The vector of the instantaneous angular velocity ω_2 of the outer ring is directed along the $O\zeta$ axis; its projection on the $O\zeta$ axis (its scalar component) equals ψ'. Let us assume further that the axes x, y, z are the principal axes of inertia of the gyroscope and of the inner ring.

Let $A = B$, C, be the principal moments of the inertia of the gyroscope. Let A_1, B_1, C_1 be the principal moments of inertia of the inner gimbal ring, and let the moment of inertia of the outer ring with respect to $O\zeta$ axis be A_2.

The kinetic energy of the gyroscope T, and the kinetic energies of the inner and outer rings T_1 and T_2 could be expressed as follows:

$$\begin{aligned} T &= \frac{1}{2}\left[A(\theta'^2 + \psi'^2 \sin^2\theta) + C(\varphi' + \psi' \cos\theta)^2\right] \\ T_1 &= \frac{1}{2}\left[A_1\theta'^2 + B_1\psi'^2 \sin^2\theta + C_1\psi'^2 \cos^2\theta\right] \\ T_2 &= \frac{1}{2} A_2\psi'^2 \end{aligned} \tag{1.2}$$

Let the coordinates of the center of gravity of the gyroscope and of the inner ring be (O, O, z_0), and let the weight of the gyroscope and of the inner ring be P. It is easily seen that the force function is $U = -Pz_0 \cos\theta$; and for a balanced gyroscope $z_0 = 0$.

Let us initially consider the case when the frictional forces on the gimbal axes are absent. In this case we have only gravitational forces acting on the system, its equations of motion could be expressed as the Lagrange equations, and in our case the Lagrangian is $L = T + T_1 + T_2 + U$.

The equations of motion are as follows:

$$
\begin{gathered}
(A + A_1)\theta'' - (A + B_1 - C_1)\psi'^2 \sin\theta\cos\theta + C(\varphi' + \psi'\cos\theta)\psi'\sin\theta - Pz_0\sin\theta = 0 \\
\frac{d}{dt}\left[(A + B_1)\psi'\sin^2\theta + C(\varphi' + \psi'\cos\theta)\cos\theta + C_1\psi'\cos^2\theta + A_2\psi'\right] = 0 \qquad (1.3) \\
C\frac{d}{dt}(\varphi' + \psi'\cos\theta) = 0
\end{gathered}
$$

The first integrals of the equations of motion are immediately obtained as (1.4)

$$
(A + B_1)\psi'\sin^2\theta + C(\varphi' + \psi'\cos\theta)\cos\theta + C_1\psi'\cos^2\theta + A_2\psi' = k, \quad \varphi' + \psi'\cos\theta = r_0
$$

These two integrals were taken with respect to the cyclic coordinates ψ and ϕ respectively. If we multiply the three equations (1.3) by θ', ψ', ϕ' respectively and add them, we obtain one more integral, the energy integral

$$
\begin{gathered}
(A + A_1)\theta'^2 + (A + B_1)\psi'^2\sin^2\theta + C_1\psi'^2\cos^2\theta + C(\varphi' + \psi'\cos\theta)^2 + \\
+ A_2\psi'^2 + 2Pz_0\cos\theta = h \qquad (1.5)
\end{gathered}
$$

The existence of this integral is certain because the applied force has a potential and because the constraints are independent of time.

We shall mention that the equations of motion (1.3) could be obtained in a different way. As it is customary in the theory of gyroscope [1] we may utilize the theorem of angular momentum and the following sets of equations; the moments of the whole system with respect to the $O\zeta$ axis, the moments of the gyroscope and of the inner ring with respect to the Ox axis, and the moments of the gyroscope with respect to the Oz axis.

2. The four expressions

$$
\theta = \theta_0, \qquad \theta' = 0, \qquad \psi' = \Omega, \qquad r = \omega \qquad (2.1)
$$

are particular solutions of the equations of motion (1.3) when the constants θ_0, Ω, ω satisfy the relation

$$
[(A + B_1 - C_1)\Omega^2\cos\theta_0 - C\omega\Omega + Pz_0]\sin\theta_0 = 0 \qquad (2.2)
$$

When $\theta_0 \neq 0, \pi$, the motion described by the particular solution (2.1) represents the regular precession of a gyroscope. We regard this motion as unperturbed and we shall investigate its stability with respect to the variables θ, θ', ψ' and r.

For the perturbed motion we shall substitute the following expressions:

$$
\theta = \theta_0 + \eta, \qquad \theta' = \eta' = \xi_1, \qquad \psi' = \Omega + \xi_2, \qquad r = \omega + \xi_3 \qquad (2.3)
$$

in the equations (1.3).

Equations of the perturbed motion obtained from this substitution admit the following first integrals (including only first and second order terms):

$$V_1 = (A + A_1)\xi_1^2 + [(A + B_1 - C_1)\Omega^2(\cos^2\theta_0 - \sin^2\theta_0) - Pz_0\cos\theta_0]\eta^2 +$$
$$+ [(A + B_1)\sin^2\theta_0 + C_1\cos^2\theta_0 + A_2](\xi_2^2 + 2\Omega\xi_2) + 4(A + B_1 - C_1)\Omega\sin\theta_0\cos\theta_0\,\xi_2\eta +$$
$$+ C(\xi_3^2 + 2\omega\xi_3) + 2[(A + B_1 - C_1)\Omega^2\cos\theta_0 - Pz_0]\sin\theta_0\eta + \ldots = \text{const}$$

$$V_2 = [(A + B_1 - C_1)\Omega(\cos^2\theta_0 - \sin^2\theta_0) - \tfrac{1}{2}C\omega\cos\theta_0]\eta^2 +$$
$$+ [2(A + B_1 - C_1)\Omega\cos\theta_0 - C\omega]\sin\theta_0\eta + [(A + B_1)\sin^2\theta_0 + C_1\cos^2\theta_0 + A_2]\xi_2 +$$
$$+ 2(A + B_1 - C_1)\sin\theta_0\cos\theta_0\xi_2\eta + C\xi_3(\cos\theta_0 - \sin\theta_0\eta) + \ldots = \text{const}$$

$$V_3 = \xi_3 = \text{const} \qquad (2.4)$$

The rows of dots indicate the omitted terms of higher order.

We shall now construct the Liapunov function in the form of the linear combination of the integrals (2.4):

$$V = V_1 - 2\Omega V_2 + 2C(\Omega\cos\theta_0 - \omega)V_3 + \frac{C^2}{A + B_1 - C_1}V_3^2 = (A + A_1)\xi_1^2 +$$
$$+ [(A + B_1)\sin^2\theta_0 + C_1\cos^2\theta_0 + A_2]\xi_2^2 + C\left(1 + \frac{C}{A + B_1 - C_1}\right)\xi_3^2 -$$
$$- [(A + B_1 - C_1)(\cos^2\theta_0 - \sin^2\theta_0)\Omega^2 - C\Omega\omega\cos\theta_0 + Pz_0\cos\theta_0]\eta^2 + 2C\Omega\sin\theta_0\eta\xi_3 + \ldots \qquad (2.5)$$

The function $V(\xi_1, \xi_2, \xi_3, \eta)$ is a positive-definite function of its arguments when the following single condition is satisfied:

$$(A + B_1 - C_1)(\cos^2\theta_0 - \sin^2\theta_0)\Omega^2 - C\omega\Omega\cos\theta_0 + Pz_0\cos\theta_0 < 0 \qquad (2.6)$$

According to the Liapunov stability theorem this condition is a sufficient condition for the stability of the regular precession (2.1) of the gyroscope on gimbals. When $\sin\theta_0 \neq 0$, $\Omega \neq 0$, then using the equation (2.2), we can express the above condition in the form

$$A + B_1 - C_1 > 0 \qquad (2.7)$$

When the condition (2.7) is satisfied, the regular precession of a gyroscope on gimbals is stable with respect to the variables θ, θ', ψ', and r; hence it must be stable with respect to the variables θ, p, q, and r. The obtained result is obviously valid for $z_0 \neq 0$ as well as for $z_0 = 0$.

We shall investigate further the case $\theta_0 = 0$, when the unperturbed motion consists of a uniform rotation of the outer ring about the vertical axis with angular velocity Ω, and a uniform rotation of the gyroscope with angular velocity ω. In this case the condition (2.2) is satisfied for arbitrary values of Ω and ω. A sufficient condition for stability of such a motion follows from (2.6) and is [2]

$$(A + B_1 - C_1)\,\Omega^2 - C\omega\Omega + Pz_0 < 0 \qquad (2.8)$$

It is obvious that if the above inequality is satisfied, then simultaneously the conditions

$$C^2\omega^2 - 4\,(A + B_1 - C_1)\,Pz_0 > 0, \qquad \Omega_1 < \Omega < \Omega_2 \qquad (2.9)$$

must also be satisfied. Here Ω_1, and Ω_2, are roots of the polynomial

$$(A + B_1 - C_1)\Omega^2 - C\omega\Omega + Pz_0 = 0$$

When $B_1 = C_1 = 0$, then the first inequality (2.9) reduces to the condition of Maievski [3]. The Maievski condition is the necessary and sufficient condition for the stability of the Lagrange gyroscope. The second inequality in (2.9) would remain as it is.

We shall prove now that the condition (2.8) is also a necessary condition for the stability of rotation about the vertical axis of a gyroscope on gimbals.

Consider the function:

$$V = (A + A_1)\,\eta\eta' \qquad (2.10)$$

and its time derivative taking into account the perturbed motion,

$$V' = (A + A_1)\,\eta'^2 + [(A + B_1 - C_1)\,\Omega^2 - C\omega + Pz_0]\eta^2 + \ldots$$

The function V' would be positive-definite if the following condition is satisfied:

$$(A + B_1 - C_1)\,\Omega^2 - C\omega\Omega + Pz_0 > 0$$

The function V admits an infinitely small upper bound and could assume positive values. On the strength of the Liapunov theorem the motion of a gyroscope would be unstable with respect to the variables η, and η' under such a condition.

It follows that the condition (2.8) must be necessary and sufficient for the stability of rotation about the vertical axis of a gyroscope on gimbals.

Without any loss of generality we can take $\omega > 0$. When $z_0 > 0$ the quantities Ω_1, and Ω_2 are positive when $A + B_1 - C_1 > 0$, which we shall assume to be the case. When $\Omega < \Omega_1$, and in the special cases when $\Omega = 0$, or when $\Omega > \Omega_2$, the motion of a gyroscope about the vertical axis would be unstable in spite of the fact that the first condition in (2.9) is satisfied. When $z_0 < 0$, the first condition in (2.9) is satisfied for any angular velocity ω, and in this case $\Omega_1 < 0$ and $\Omega_2 > 0$. When the angular velocity of precession Ω satisfies one of the inequalities $\Omega < \Omega_1$ or $\Omega > \Omega_2$, the motion is unstable.

For a balanced gyroscope ($z_0 = 0$) the necessary and sufficient condition (2.8) for the stability of rotation about the vertical axis assumes the following form [1]

$$0<\Omega<\frac{C\omega}{A+B_1-C_1} \tag{2.11}$$

It should be mentioned that the problem of stability of a gyroscope on gimbals could be analyzed also by the application of the Routh theorem Solving the cyclic integrals (1.4) for ψ' and ϕ' we obtain

$$\psi'=\frac{k-Cr_0\cos\theta}{(A+B_1)\sin^2\theta+C_1\cos^2\theta+A_2}, \qquad \varphi'=r_0-\psi'\cos\theta$$

and we construct the Routh function

$$R=L-Cr_0\varphi'-k\psi'=\frac{1}{2}(A+A_1)\theta'^2+W(\theta)$$

where the new force function is

$$W(\theta)=-\left(Pz_0\cos\theta+\frac{1}{2}\frac{(k-Cr_0\cos\theta)^2}{(A+B_1)\sin^2\theta+C_1\cos^2\theta+A_2}+\frac{1}{2}Cr_0^2\right) \tag{2.12}$$

The function R is independent of the time t, hence the first integral of the Routh equation corresponds to the integral of kinetic energies:

$$H=\frac{\partial R}{\partial\theta'}\theta'-R=\frac{1}{2}(A+A_1)\theta'^2-W=h$$

The first integral of the perturbed equation is

$$\frac{1}{2}(A+A_1)\eta'^2-W(\theta_0+\eta)+W(\theta_0)=\text{const}$$

with the condition that the constants r_0 and k remain unperturbed. On the strength of the Routh theorem the motion is stable with respect to θ and θ' when for the unperturbed motion the new force function W has a maximum For example, it could be easily shown when $\theta_0 = 0$, that the condition for a maximum of the function W has the form of the inequality (2.8).

3. Beside gravitational forces we shall admit now frictional forces acting on a gyroscope on gimbals. With frictional forces present, the equations of motion will differ from the equations (1.3) in that their right-hand members will have the corresponding moments of the resistance forces. These equations would admit the following particular solution

$$\theta=0, \qquad \theta'=0, \qquad \psi'=\Omega, \qquad \varphi'=\omega_1 \qquad (\omega_1=\omega-\Omega) \tag{3.1}$$

only on the condition that the moments of the supplementary forces about the axes $O\zeta$ and Oz are applied to the system in order to equilibrate the moments due to frictional forces. Let us investigate the stability of the motion described by (3.1) by setting in the perturbed motion

$$\theta=\eta, \qquad \theta'=\eta'=\xi_1, \qquad \psi'=\Omega+\xi_2, \qquad \varphi'=\omega_1+\zeta$$

Let us assume besides that in the perturbed motion the dissipative forces acting on a gyroscope on gimbals are time derivatives of the Rayleigh function

$$2f = a\xi_1^2 + b\xi_2^2 + c\zeta^2 + 2e\xi_2\zeta + 2f\zeta\xi_1 + 2g\xi_1\xi_2$$

which is a positive-definite quadratic form of ξ_1, ξ_2, ζ.

The variational equations for the perturbed motion are as follows:

$$\begin{gathered}(A + A_1)\xi_1' - [(A + B_1 - C_1)\Omega^2 - C\omega\Omega + Pz_0]\eta = -(a\xi_1 + g\xi_2 + f\zeta) \\ (A_2 + C + C_1)\xi_2' + C\zeta' = -(g\xi_1 + b\xi_2 + e\zeta) \\ C(\zeta' + \xi_2') = -(f\xi_1 + e\xi_2 + c\zeta)\end{gathered} \tag{3.2}$$

Let us consider the function

$$\begin{gathered}2W = (A + A_1)\xi_1^2 + (A_2 + C + C_1)\xi_2^2 - [(A + B_1 - C_1)\Omega^2 - C\omega\Omega + Pz_0]\eta^2 + \\ + C\zeta^2 + 2C\xi_2\zeta + 2\varepsilon(A + A_1)\eta\xi_1\end{gathered} \tag{3.3}$$

where ϵ is a certain positive constant. On the strength of (3.2) the time derivative of the above function equals

$$\begin{gathered}W' = -\{[a - \varepsilon(A + A_1)]\xi_1^2 + b\xi_2^2 + c\zeta^2 + 2e\xi_2\zeta + 2f\xi_1\zeta + 2g\xi_1\xi_2 - \\ - \varepsilon[(A + B_1 - C_1)\Omega^2 - C\omega\Omega + Pz_0]\eta^2 + \varepsilon\eta(a\xi_1 + g\xi_2 + f\zeta)\}\end{gathered} \tag{3.4}$$

It is obvious that in order to make the function W positive-definite and its time derivatives W' negative-definite with respect to the variables ξ_1, ξ_2, ζ, η, under the somewhat stronger condition (2.8) (in the right member of (2.8) zero is replaced by $\delta(\epsilon) < 0$) we must select a sufficiently small constant ϵ. If we do this, then the function W would satisfy the conditions of the Liapunov theorem on the asymptotic stablity. The stable motion (3.1) becomes asymptotically stable under the action of dissipative forces. The above statement is valid also in the case when the masses of the gimbal rings are neglected, that is when

$$A_1 = B_1 = C_1 = A_2 = 0$$

Lord Kelvin proposed [3] to call the stability of equilibrium arising from a gyroscopic stabilization "the temporary stability", and the stability resulting from the conservative forces "the secular stability". Extending the last definition to the case of a steady motion we conclude that the motion (3.1) of a gyroscope on gimbals under the action of gravitational forces and the constant moments of the supplementary forces is stable in the secular sense. This circumstance is closely connected with the fact that for the motion (3.1) the new force function (2.12) has a maximum.

BIBLIOGRAPHY

1. Nikolai, E.L., O dvizheni uravnoveshennogo giroskopa v kardanovom podvese (On the motion of a balanced gyroscope on gimbals). *PMM* Vol. 3, No. 4, 1939.

2. **Magnus, K.**, Ob ustoichivosti dvizhenia tiazhelogo simmetricheskogo giroskopa v kardanovom podvese (On the stability of motion of a heavy symmetrical gyroscope on gimbals). *PMM* Vol. 22, No. 2, 1958.

3. Chetaev, N.G., *Ustoichivost dvizhenia* (*Stability of Motion*). Gostekhizdat, 1955.

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